

## Chapter 14

# Uniform Acceleration

### 14.1 Events at the same proper distance from some event

Consider the set of events that are at a fixed proper distance from some event. Locating the origin of space-time at this event, the equation for this set of events is:

$$x^2 - c^2t^2 = d^2 \tag{14.1}$$

The parameter,  $d$ , is the proper distance of these events from the origin event. The origin event and the events on the curve are related by this distance  $d$  and thus for the set of events on the curve the origin is called the magic event and  $d$  is the distance from the magic event to the curve.

In space-time, this is a two branch hyperbola with light cones emanating from the origin as the asymptotes. If we now consider only the branch that has  $x > 0$ ,  $x = \sqrt{d^2 + c^2t^2}$ , we have a single curve. In Figure 14.2 on page 311, We plot several of them for different  $d$ . Since this equation is a form invariant under the Lorentz transformations, all inertial observers will have the same curve and Lorentz transformations will map points on the curve to points on the curve.

By locating a light cone on the event at  $(d, 0)$ , we can see that all the events on the curve at later times are in the future; the curve is monotonically asymptotic to a light cone that is later in space-time. Thus all the events at later times on the curve are in the future of  $(d, 0)$ . Similarly all the events that are before  $t = 0$  are in the past of  $(d, 0)$ . Thus the curve is time-like and is therefore a candidate for the motion of a material particle. In the next section, we will see that this is the trajectory of the uniformly accelerated object.

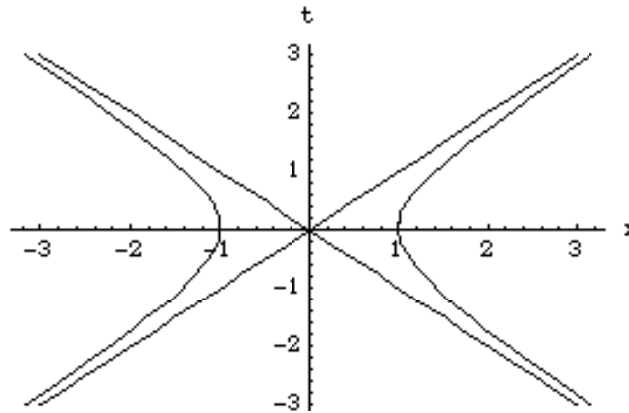


Figure 14.1: The locus of events that are at the same proper distance from the origin. These curves all points of which are elsewhere related to the origin are pairwise time-like related.

## 14.2 Uniformly accelerated motion

Since this curve is time-like, it is a possible state of motion for a material particle. It is certainly a case of motion that is not uniform, not a straight line in space-time. For any observer in uniform motion, an object following this trajectory will appear to be approaching at a very rapid rate, almost  $-c$ , and slowing down until at some event it is as close as it will ever get and at rest with respect to the observer and then moving away so that at long times later it is receding at almost  $c$ .

Since the Lorentz transformations are homogeneous and linear, lines through the origin are transformed into lines through the origin and space-like lines are transformed into space-like lines and similarly for time-like lines. Thus if you pick an event, say  $(x_0, t_0)$ , on this curve, the line through it and the origin which is space-like can be transformed to the space-like line through  $(d, 0)$  and  $(0, 0)$  by the Lorentz transformation with  $v = c^2 \frac{t_0}{x_0}$ . This is also the transformation that brings the tangent to the curve to the vertical which means that the instantaneous relative velocity at  $(x_0, t_0)$  is  $v$ . Or said another way, an observer with relative velocity,  $v = c^2 \frac{t_0}{x_0}$ , is a comover to the this trajectory at the event  $(x_0, t_0)$ . Thus we see that the instantaneous relative velocity at  $(x_0, t_0)$  is  $v = c^2 \frac{t_0}{x_0}$ . More significantly, to the respective comovers, the acceleration at  $(x_0, t_0)$  is the same as the acceleration at  $(d, 0)$ . Therefore, as measured by comovers, the instantaneous acceleration at any event is the same and this is the acceleration that the object experiences in

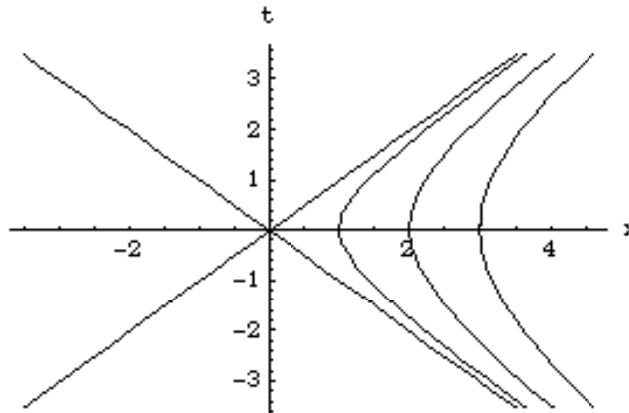


Figure 14.2: The locus of events with  $x > 0$  that are at the same proper distance from the origin for different values of the proper distance,  $d$ .

its motion. On simple dimensional grounds, the acceleration at the event  $(d, 0)$  must be

$$a = \frac{c^2}{d}. \quad (14.2)$$

Also note that it follows from the previous argument that the line from  $(x_0, t_0)$  to the origin is the line of simultaneity for the commover at the event  $(x_0, t_0)$ .

### 14.2.1 Details of the calculation of the acceleration

The easiest way to calculate the acceleration is to use calculus. For a uniformly accelerated curve the instantaneous co-mover's velocity is

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt}(\sqrt{d^2 + c^2 t^2}) \\ &= \frac{1}{2} \times \frac{2c^2 t}{\sqrt{d^2 + c^2 t^2}} \\ &= c^2 \frac{t}{x} \end{aligned} \quad (14.3)$$

which we already knew. The acceleration is

$$\frac{d^2 x}{dt^2} = \frac{d(c^2 \frac{t}{x})}{dt}$$

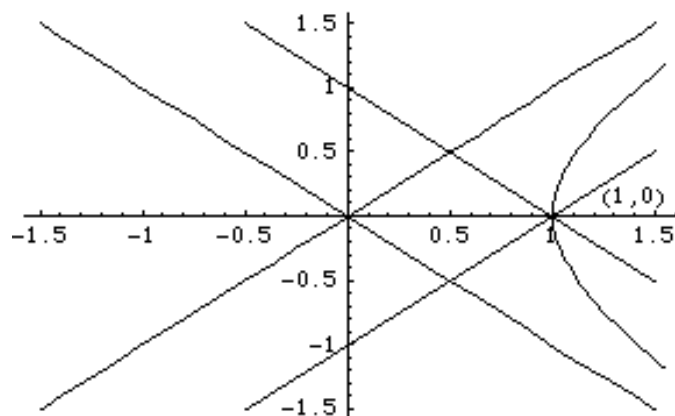


Figure 14.3: Placing a light cone at the event  $(1, 0)$  shows that the locus of events with  $x > 0$  that are at the same proper distance,  $d = 1$ , from the origin is a timelike trajectory.

$$\begin{aligned}
 &= c^2 \left( \frac{1}{x} - \frac{t}{x^2} \frac{dx}{dt} \right) \\
 &= c^2 \left( \frac{1}{x} - c^2 \frac{t}{x^2} \times \frac{t}{x} \right) \\
 &= c^2 \left( \frac{x^2 - c^2 t^2}{x^3} \right) \\
 &= c^2 \frac{d^2}{x^3}
 \end{aligned} \tag{14.4}$$

which, at the event  $(d, 0)$ , means that  $v = 0$  and  $a = \frac{c^2}{d}$ , which was our result from dimensional arguments in Equation 14.2 on page 311.

If you have an aversion to calculus, you can look at the motion for small times near the event  $(d, 0)$ . It must reduce to the expression for the position for the constant acceleration that we know from classical physics,  $x_{cl}(t) = x_0 + v_0 t + \frac{a}{2} t^2$  which should be valid for  $\frac{at}{c} \ll 1$ . Expanding our  $x(t)$  for small  $t$  and using the fact,  $(1 + x)^n \approx 1 + nx$  for  $x \ll 1$ , that everyone should know from Section 1.4.2 on page 16, we have

$$\begin{aligned}
 x(t) &= \sqrt{d^2 + c^2 t^2} \\
 &= d \sqrt{1 + \frac{c^2}{d^2} t^2} \\
 &\approx d \left( 1 + \frac{c^2}{2d^2} t^2 \right).
 \end{aligned} \tag{14.5}$$

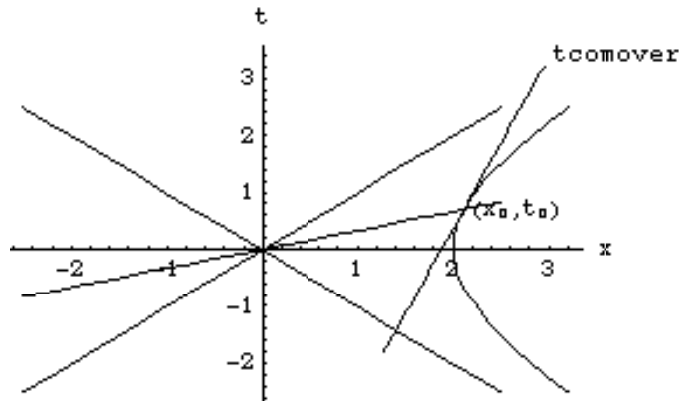


Figure 14.4: The uniformly accelerated observer with the world line and the line of simultaneity of the commover for the event  $(x_0, t_0)$ .

Comparing this with  $x_{cl}(t)$ , we see that, for small times near the event  $(d, 0)$ , the velocity is 0 and the acceleration is  $\frac{c^2}{d}$ , again our result Equation 14.2 on page 311.

It is very important to point out that this is the acceleration that the accelerated object “feels”. Consider an accelerated rocket with a pair of identical springs and masses, one mass-spring system mounted on a frictionless surface horizontally and the other mass-spring suspended vertically. Vertical in the rocket is along the line from front to back and horizontal is one of the transverse directions. We also calibrate our springs so that we know the force that is required to stretch them a given amount, i. e. we know the spring constant,  $k$ , of the springs. The horizontal mass-spring will have one equilibrium position and the vertical one will have a different one. If we now carefully adjust the thrust of the rocket so that the stretch of the springs does not change with time, our rocket when observed by someone who was initially at rest with us will register it at  $x(t) = \sqrt{d^2 + c^2 t^2} - d$  where  $d = \frac{c^2}{\frac{k \times \text{stretch}}{m}}$  where  $m$  is the mass,  $k$  is the calibrated spring constant, “stretch” is the difference in the length of the vertical and horizontal springs. The extra  $d$  is in  $x(t)$  to make the rocket and the original commover coincident in space at  $t = 0$ . At later times, the rocket has moved away from the original commover but the mass-spring system still measures the same acceleration, the acceleration that is measured by the new instantaneous commover.

This is another case of a term which is dimensionally the same but whose

physical interpretation is different. Acceleration is generally defined kinematically as  $a_k \equiv \frac{d^2x(t)}{dt^2}$ . Through Newton's laws, we have an equivalent definition in the form  $a_s \equiv \frac{f}{m}$  where  $f$  is the effect of external objects on a body of mass  $m$ . It is this  $a_s$  that is "sensed" by the accelerated system that informs it that it is not inertial. This is the essence of Galilean invariance. A free body has no acceleration. The equality of  $a_s$  and  $a_k$  expressed in Newton's law can be required only in the case of a world of low relative velocities. Since the kinematic definition is not a constant in this motion although the sensed acceleration is constant, we have an interpretation problem. It is required that all inertial observers of this motion agree on its sensed acceleration and from the previous discussion all events on the trajectory have the same sensed acceleration to a local commover and this acceleration is the same as the kinematic one as evaluated or measured for small times around the events when the object is commoving with that observer. For all other times, the kinematic and sensed acceleration are different. The kinematic acceleration is the acceleration evaluated by one of the commover inertial observers for all time and it varies from  $\frac{c^2}{d}$  the small time value to zero at large times when the object is distant. The kinematic acceleration is  $\frac{d^2}{dt^2}x(t)$  where both  $x$  and  $t$  are coordinates for the specific inertial commover. An alternative might be to call this motion not uniformly accelerated motion but uniformly effected motion.

### 14.3 The proper time along the trajectory

As was stated in Section 12.3 on page 280, the proper time between two events is a trajectory dependent concept. As the accelerated object moves along its trajectory, its coordinate position and time are given by  $x(t) = \sqrt{d^2 + c^2t^2}$ . This same motion can be conceived of as both  $x$  and  $t$  both evolving as a function of the proper time,  $x(\tau)$  and  $t(\tau)$ . Our problem is to find these relationships. Noting that because of the definition of the trajectory as the locus of events with the same proper distance from the origin event that for all  $\tau$  the two functions  $x(\tau)$  and  $t(\tau)$  satisfy  $(x(\tau))^2 - c^2(t(\tau))^2 = d^2$ .

#### 14.3.1 Timelike Trajectories and Accelerated Motion

Although it does not constitute a proof, we can use accelerated motion to justify the often heard comment that there is no force that can boost a material particle to speeds greater than the speed of light. As stated in

Section 14.2.1 on page 311, the acceleration  $a$  that labels this trajectory is the acceleration that a material particle moving along that world line “feels.” In other words, the force that accelerates the particle to move it along this trajectory is a constant as measured by the sequence of comovers and these are the suitable observers of the force of acceleration. In this case of constant force, we see that no matter how long the force operates, the velocity of the particle that is subjected to this force moves relative to its initial velocity at a speed that is less than  $c$ ; the trajectory remains timelike for all times. Also in any finite time interval, there is no acceleration and thus no force that can change the trajectory from timelike to space like.

## 14.4 Examples using accelerated motion

With the tools developed in the previous sections, we can now analyze all kinds of simple uniform acceleration problems. In fact, just about any of the usual uniform acceleration problems that are encountered classical physics can be studied. In this section, I will go through the details of three typical problem types.

### 14.4.1 Deceleration

Sally is moving toward a wall with a relative speed of  $\frac{3}{5}c$ . When she is one lightyear away from the wall, she decides to decelerate. What is the minimum deceleration that she can use so that she just comes to rest at the wall?

We can find the answer in the frame in which the wall is at rest. Firstly, we should diagram the motion.

From this we can see that the problem can be stated in a simpler fashion. At any event,  $(x_0, t_0)$ , on the uniformly accelerated trajectory, we know the relative velocity at that point,  $\frac{v}{c^2} = \frac{t_0}{x_0}$ . For the case shown in Figure 14.5 on page 316, note that  $t_0$  is negative and  $x_0$  is positive so that  $v$  is negative. Thus we can ask given an acceleration,  $a$ , how far from the event  $(x_0, t_0)$  on that trajectory is the vertex of the hyperbola? In the usual coordinate system, the vertex is at  $(d, 0)$  and thus the stopping distance for that case is  $\delta = x_0 - d$ . Remember the  $d$  is related to the acceleration,  $a$ , as  $d = \frac{c^2}{a}$ . The event  $(x_0, t_0)$  satisfies  $t_0 = \frac{v}{c^2}x_0$  where  $v$  is the relative velocity at that event and  $x_0 = \sqrt{d^2 + c^2t_0^2}$  or  $x_0 = \frac{d}{\sqrt{1-\frac{v^2}{c^2}}}$ . Thus the general formula for

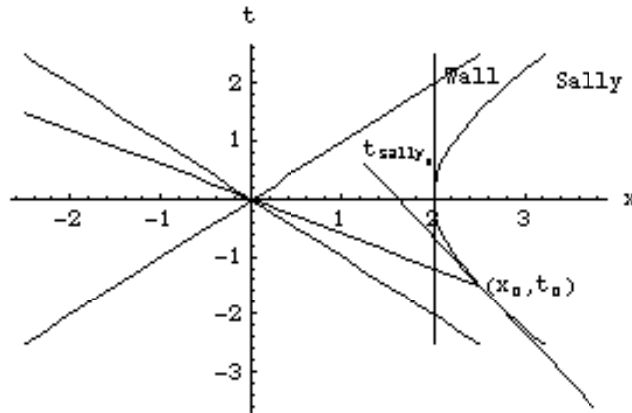


Figure 14.5: Sally turning from the wall. The event  $(x_0, t_0)$  is the event at which she decelerates. The line labeled “Sally” is her trajectory. The line labeled “ $t_{\text{sally}_0}$ ” is her worldline before decelerating

the stopping distance for a given velocity and acceleration is

$$\delta = d \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) = \frac{c^2}{a} \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right).$$

The next problem is to decide what  $\delta$  is. From the problem setting, I would argue that the one light year distance is the coordinate distance in her frame at the instant that she starts the acceleration. This  $\delta$  is the distance in the wall’s frame. This is not Sally’s distance. That distance is the proper distance between the event  $(x_0, t_0)$  and the intersection of the line of simultaneity of the commover at  $(x_0, t_0)$  and the worldline of the wall. The equation of the line of simultaneity is  $\frac{t-t_0}{x-x_0} = \frac{v}{c^2}$  and the line of the wall is  $x = d$ . The event at the intersection of these two lines is  $(d, \frac{v}{c^2}(d-x_0) + t_0)$  and the proper distance between this event and the event at the start of the acceleration,  $(x_0, t_0)$ , is  $\sqrt{1 - \frac{v^2}{c^2}}(x_0 - d)$ . Calling her distance to the wall  $\delta'$ , we now have  $a = \frac{c^2}{\delta'} \left( 1 - \sqrt{1 - \frac{v^2}{c^2}} \right)$ .

How does this compare to the classical result, stopping distance =  $\frac{v^2}{2a}$ ? From “Things”, Section 1.4.2 on page 16, for large  $c$ ,  $\sqrt{1 - \frac{v^2}{c^2}} \approx 1 - \frac{v^2}{2c^2}$ . Plugging this in we have the classical result exactly.

For our specific problem, we have  $v = -\frac{3}{5}c$  and  $\delta' = 1$  ltyr and  $a = \frac{1}{5} \frac{\text{ltyr}}{\text{yr}^2}$  or  $2 \frac{\text{m}}{\text{s}^2}$ .

### 14.4.2 Accelerated Rocket

A rocket of length  $\frac{1}{2}$  lightyear is accelerated at a constant acceleration of  $\frac{1}{2} \frac{\text{lightyear}}{\text{year}^2}$ . At  $t = 0$ , the rocket starts to accelerate. When a clock at the bottom reads a time  $\tau_{\text{bottom}}$ , what is the time for a clock in the top of that rocket?

Again, we have to determine what is being told to us in the problem. We have to decide where the parts of the rocket are, i. e. their world lines. The top of the rocket is rigidly connected to the bottom so that as the rocket accelerates the distance as measured from the bottom of the rocket to the top is unchanged. Under stress but unchanged. The world line of the bottom which is accelerating at a rate  $a_{\text{bottom}}$  in the standard coordinate system is

$$\begin{aligned} x(\tau_{\text{bottom}}) &= \frac{c^2}{a_{\text{bottom}}} \cosh\left(\frac{a_{\text{bottom}}}{c} \tau_{\text{bottom}}\right) \\ t(\tau_{\text{bottom}}) &= \frac{c^2}{a_{\text{bottom}}} \sinh\left(\frac{a_{\text{bottom}}}{c} \tau_{\text{bottom}}\right) \end{aligned} \quad (14.6)$$

or, using  $d = \frac{c^2}{a_{\text{bottom}}}$ , where  $d$  is the proper distance from the origin event,  $(0, 0)$ , to any event on the world line,

$$\begin{aligned} x(\tau_{\text{bottom}}) &= d \cosh\left(\frac{c}{d} \tau_{\text{bottom}}\right) \\ t(\tau_{\text{bottom}}) &= d \sinh\left(\frac{c}{d} \tau_{\text{bottom}}\right). \end{aligned}$$

Note that the commover to any event,  $(x(\tau_{\text{bottom}}), t(\tau_{\text{bottom}}))$ , has a line of simultaneity that goes from that event through the origin event,  $(0, 0)$ .

A second set of events that are all at a proper distance  $d + h$  from the origin event,  $(0, 0)$ , see Figure 14.6 on page 318 would be at

$$\begin{aligned} x(\tau_{\text{top}}) &= (d + h) \cosh\left(\frac{c}{d + h} \tau_{\text{top}}\right) \\ t(\tau_{\text{top}}) &= (d + h) \sinh\left(\frac{c}{d + h} \tau_{\text{top}}\right). \end{aligned}$$

Also since the lines of simultaneity are the lines through the origin event, the distance between these world lines when measured by the commover at the bottom of the rocket is  $h$ . The trajectory of the top of the rocket is

$$\begin{aligned} x(\tau_{\text{top}}) &= \frac{c^2}{a_{\text{top}}} \cosh\left(\frac{a_{\text{top}}}{c} \tau_{\text{top}}\right) \\ t(\tau_{\text{top}}) &= \frac{c^2}{a_{\text{top}}} \sinh\left(\frac{a_{\text{top}}}{c} \tau_{\text{top}}\right) \end{aligned} \quad (14.7)$$

Thus these are the world lines of the top and the bottom of the rocket.

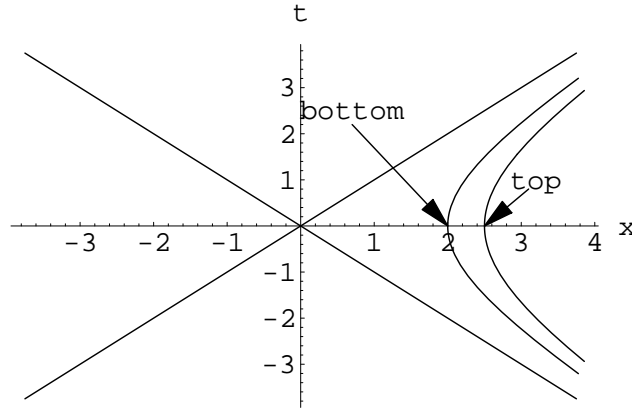


Figure 14.6: **The world lines of the top and bottom of an accelerating rocket.** The bottom of the rocket has an acceleration of  $\frac{1}{2} \frac{\text{lightyear}}{\text{year}^2}$ . The top of the rocket is at a distance  $\frac{1}{2}$  lightyear from the bottom.

We see immediately that the top of the rocket does not have the same acceleration as the bottom. Using  $d = \frac{c^2}{a_{\text{bottom}}}$ , we get that

$$a_{\text{top}} = \frac{a_{\text{bottom}}}{1 + \frac{ha_{\text{bottom}}}{c^2}}. \quad (14.8)$$

In Figure 14.6 on page 318, we also see that, since the world lines of the top and the bottom of the rocket share the same asymptotes, the hangle to the line of simultaneity to any event is the same and thus that

$$\phi = \frac{c\tau_{\text{bottom}}}{d} = \frac{c\tau_{\text{top}}}{d+h}$$

or writing this in terms of the accelerations of the rocket,

$$\phi = \frac{a_{\text{bottom}}\tau_{\text{bottom}}}{c^2} = \frac{a_{\text{top}}\tau_{\text{top}}}{c^2}$$

or

$$\tau_{\text{top}} = \left(1 + \frac{ha_{\text{bottom}}}{c^2}\right)\tau_{\text{bottom}}. \quad (14.9)$$

Thus, clocks at the top and bottom of a rocket run at different rates. This situation can be made a little more baffling by noting that although the top and bottom of the rocket have clocks that run at different rates, the top and bottom share the same lines of simultaneity. They just differ about the time of these simultaneous events.

### 14.4.3 John Bell's Problem

The next example is the problem of two identical rockets and John Bell's Problem. Although I am not able to vouch for this story directly, I have been told the following fascinating story about John Bell. Yes, the same John Bell of Bell's Theorem, see Chapter 21 on page 471. When a new theoretical physicist would come to the world famous laboratory, CERN, where Bell was employed, Bell would go to lunch room and look up the new person and, as a part of the getting-to-know-you chit chat, ask the new person the following question: If two identical coasting rockets were connected by a string and the rockets then given identical uniform accelerations would the string between them break after some time?

Without making a careful analysis, usually without even thinking about it carefully, the unsuspecting innocent would quickly answer that the string would not break. The quick argument being that, if the two rockets were moving at the same velocity originally and had identical accelerations, they would always stay the same distance apart. We are now enough informed about the interesting effects of relativity and particularly uniform acceleration in special relativity to be a little more careful. If identical clocks at the top and bottom of a rocket can drift apart in time, then it is plausible that identical rockets can begin to separate, see Section 14.4.2 on page 317 above. The proof that the string will break is easily shown graphically, see Figure 14.7 on page 320.

Well, at least in principle, it is simple even if the figure is rather complex. Two identically uniformly accelerated rockets have trajectories that are shifted from each other. Consider two rockets that are separated by a distance  $h$  and have an acceleration  $a$ , their trajectories are

$$\begin{aligned} x_{top} &= \sqrt{c^2 t^2 + \left(\frac{c^2}{a}\right)^2}, \\ x_{bottom} &= \sqrt{c^2 t^2 + \left(\frac{c^2}{a}\right)^2} - h, \end{aligned} \quad (14.10)$$

where the top rocket is the one to the side of the acceleration.

The end of a string of length  $h$  suspended from the top rocket has the trajectory

$$x_{string} = \sqrt{c^2 t^2 + \left(\frac{c^2}{a} - h\right)^2}. \quad (14.11)$$

It is clear that the  $x_{string} - x_{bottom} > 0$  for all  $t$ . In fact, we can easily calculate the separation for small  $\frac{ah}{c^2}$ , physically not an unreasonable criteria

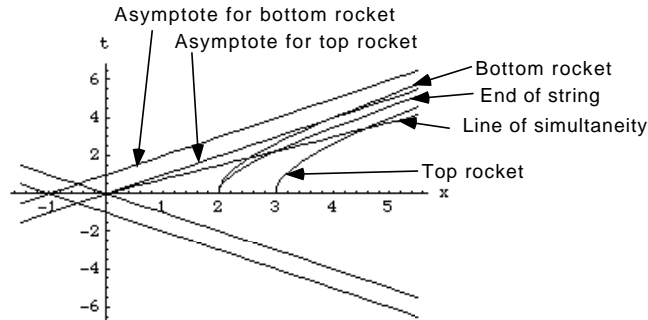


Figure 14.7: **John Bell's Problem** Two identical rockets have trajectories that follow each other. We define bottom and top as in the earlier example, Section 14.4.2 on page 317, by the direction of the acceleration. If a string is suspended from the top rocket that just reaches the bottom rocket at  $t = 0$ , it will have the trajectory shown. Since the end of the string moves so that it is a fixed distance from the top rocket as measured by the top rocket, it shares the same asymptote as the top rocket. The bottom rocket has a different asymptote and, in fact its trajectory crosses the top rockets asymptote. Thus it is clear that it is further than the end of the string from the top rocket. Since the string and top rocket share the same line of simultaneity, you can see along that line that at any time  $t$  to the top rocket the bottom rocket is further than the end of the string. The parameters for this figure were  $a = \frac{1}{3} \frac{\text{lyrs}}{\text{yr}^2}$ ,  $h = 1 \text{ ltyr}$ .

for the size of the rocket and the acceleration. In this limit and after a couple of applications of the result from “Things”, Section 1.4.2 on page 16 and some rather tedious algebra,

$$x_{string} - x_{bottom} = h \left( 1 - \frac{1}{\sqrt{1 + \frac{a^2 t^2}{c^2}}} \right). \quad (14.12)$$

Again, it is clear that this is positive for all  $t$ . The problem is that this is not the length of interest if the question is when the string will break. Equation 14.12 on page 320 is the separation of the end of the string and the bottom rocket to the original co-mover at some time  $t$  according to that inertial observer's clock. We really want the distance the string realizes at any time  $\tau$  to the string. Of course, we realize from the previous example, Section 14.4.2 on page 317, that different parts of the string have different times. Fortunately though, the elements of the string all share the same

line of simultaneity and it is, of course, the same as that of the top rocket. This quandary about clocks along accelerated systems will be examined in more detail in the Section 14.5 on page 322 where we discuss the problem of allowing an accelerated observer to create a coordinate system. It is also discussed in the development of General Relativity on the implications of the Equivalence Principle, see Section 16.4 on page 350.

Using as our time, the time  $\tau$  of the top rocket, we can determine the events at the end of the string and bottom rocket that are simultaneous with  $\tau$  on the top rocket. The equation for the line of simultaneity to the top rocket for any event,  $(x_0, t_0)$ , and the string at a time  $\tau$  on the top rocket is

$$\frac{t}{x} = \frac{t_0}{x_0} = \tanh\left(\frac{a\tau}{c}\right) \quad (14.13)$$

and the event at the end of the string simultaneous with  $\tau$  at the top rocket is

$$\begin{aligned} x_{string_\tau} &= \left(\frac{c^2}{a} - h\right) \cosh\left(\frac{a\tau}{c}\right) \\ t_{string_\tau} &= \left(\frac{c^2}{a} - h\right) \frac{\sinh\left(\frac{a\tau}{c}\right)}{c}. \end{aligned} \quad (14.14)$$

The event on the bottom rocket trajectory that is simultaneous to the string and the top rocket satisfies

$$(x_{bottom_\tau} + h)^2 - \left(\frac{c^2}{a}\right)^2 = \tanh^2\left(\frac{a\tau}{c}\right) x_{bottom_\tau}^2. \quad (14.15)$$

Of the two roots of this equation, the physically acceptable one yields

$$x_{bottom_\tau} = \left[ \sqrt{\left(\frac{c^2}{a}\right)^2 - \left(\frac{c^2}{a}\right)^2 \tanh^2\left(\frac{a\tau}{c}\right) + \tanh^2\left(\frac{a\tau}{c}\right) h^2 - h} \right] \cosh^2\left(\frac{a\tau}{c}\right) \quad (14.16)$$

with the  $t_{bottom_\tau}$  given by

$$t_{bottom_\tau} = \tanh\left(\frac{a\tau}{c}\right) \frac{x_{bottom_\tau}}{c}. \quad (14.17)$$

The stretch of the string,  $\delta$ , is the proper distance between the events at the end of the string and the bottom rocket,

$$\begin{aligned} \delta &= \sqrt{(x_{string_\tau} - x_{bottom_\tau})^2 - c^2 (t_{string_\tau} - t_{bottom_\tau})^2} \\ &= (x_{string_\tau} - x_{bottom_\tau}) \sqrt{1 - \tanh^2\left(\frac{a\tau}{c}\right)} \\ &= (x_{string_\tau} - x_{bottom_\tau}) \cosh^{-1}\left(\frac{a\tau}{c}\right). \end{aligned} \quad (14.18)$$

Plugging in for  $x_{string\tau}$  and  $x_{bottom\tau}$ , and doing considerable algebra and using the hyperbolic function identities,

$$\delta = \frac{c^2}{a} - h \left( 1 - \cosh \left( \frac{a\tau}{c} \right) \right) - \sqrt{\left( \frac{c^2}{a} \right)^2 + h^2 \sinh^2 \left( \frac{a\tau}{c} \right)} \quad (14.19)$$

Using the same parameters as in Figure 14.7 on page 320, the stretch as a function of  $\tau$  is shown in Figure 14.8 on page 322. Given an elasticity and

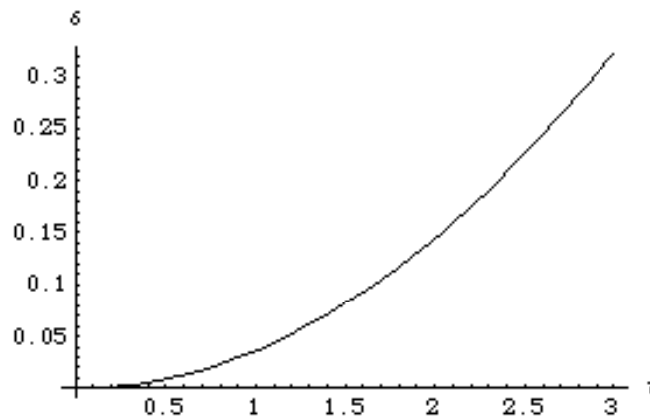


Figure 14.8: **Stretched String between Rockets** The stretch of a string connected between two identical rockets as a function of the time of the top rocket, see Figure 14.7 on page 320. The parameters for this figure were  $a = \frac{1}{3} \frac{\text{ltyrs}}{\text{yr}^2}$ ,  $h = 1 \text{ ltyr}$ .

breaking tension, we could calculate the  $\tau$  at which the string breaks but that would get us into a problem in materials engineering.

## 14.5 The Accelerated Reference Frame

Although we know that an accelerated observer does not have the same laws of physics as an inertial observer, there are often circumstances in which it is advantageous to make observations from an accelerating system. In addition, we will find that the General Theory of Relativity will have a very close and important connection with accelerated observers and the intuition that is developed here will be valuable there, see Section 16.2 on page 346.

We can proceed to construct the reference frame for an accelerating system in the same way that we did for inertial observers, see Section 11.1 on

page 241. Immediately, there are several problems. If we use the confederate procedures, i. e. placing confederates by some rule and endowing them with a clock to label events. There are actually several choices. At some time  $t$ , we could set at a fixed distance from each other a set of confederates with the same acceleration. This is not reasonable. As time goes on the confederates would find themselves drifting apart and, worst still, they would not have common lines of simultaneity, see Section 14.4.3 on page 319. Another choice would be to place them at a fixed distance but give them suitably adjusted accelerations so that they maintain their separations. In this case, all the confederates experience different accelerations, see Section 14.4.2 on page 317. Not only do they experience different accelerations, If we endow them with identical clocks, these clocks will run at different rates, again see Section 14.4.2 on page 317. Of course, we can see that since they share the same magic event, they will agree on simultaneity. Thus

$$\begin{aligned}x_{h,\tau'} &= \left(\frac{c^2}{g} + h\right) \cosh\left(\frac{g\tau'}{c}\right) - \frac{c^2}{g} \\ct_{h,\tau'} &= \left(\frac{c^2}{g} + h\right) \sinh\left(\frac{g\tau'}{c}\right)\end{aligned}\quad (14.20)$$

could be used to label events where  $(x_{h,\tau}, t_{h,\tau})$  are the event labels provided by the inertial commover of the origin confederate. In Equation 14.20 on page 323,  $h$  designates a position of the confederate and  $\tau'$  is the time on that clock.  $g$  is the acceleration of the confederate at the origin. These expressions are simplified if we refer all clock readings to the origin confederate's time, i. e. the nearest confederate records the event time on their clock and then translates to the origin confederate's time using Equation 14.9 on page 318. This implies that one of the origin confederate plays a special role and is "in charge." With this change, we have

$$\begin{aligned}x_{h,\tau} &= \left(\frac{c^2}{g} + h\right) \cosh\left(\frac{g\tau}{c}\right) - \frac{c^2}{g} \\ct_{h,\tau} &= \left(\frac{c^2}{g} + h\right) \sinh\left(\frac{g\tau}{c}\right)\end{aligned}\quad (14.21)$$

We can invert this system to yield the equations of  $h$  and  $\tau$  in terms of the inertial coordinate labels,

$$h = \sqrt{\left(x_{h,\tau} + \frac{c^2}{g}\right)^2 - c^2 t_{h,\tau}^2} - \frac{c^2}{g}$$

$$\tau = \frac{c}{g} \tanh^{-1} \left( \frac{ct_{h,\tau}}{x_{h,\tau} + \frac{c^2}{g}} \right) \quad (14.22)$$

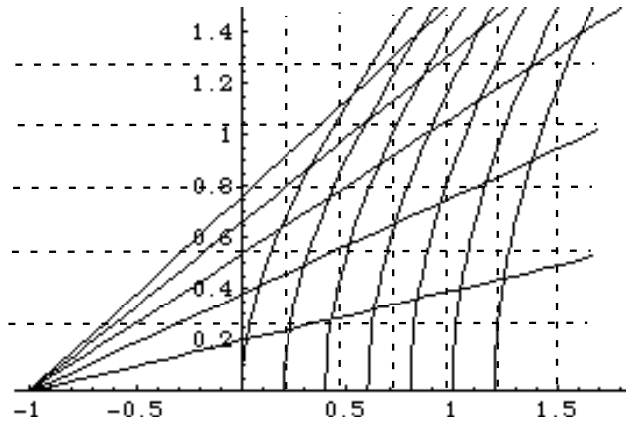


Figure 14.9: **Coordinate grid for a uniformly accelerated observer by means of confederates.** The time-like world line passing through the origin event is that of an observer that has an acceleration of  $1 \frac{\text{lyr}}{\text{yr}^2}$ . This is the reference observer for this coordinate system composed of confederates at fixed distances from the reference observer. The space-like lines are the locus of events coordinatized at the same time in this coordinate system. Shown dotted are the lines of constant time and place as determined by an inertial observer that is comoving with the accelerated observer at the initial event.

This coordinate scheme still has very serious drawbacks. The farthest confederate below the reference observer is at the magic event,  $h = -\frac{c^2}{g}$  and that confederate has an infinite acceleration. The range in  $\tau$  is  $-\infty < \tau < \infty$ . In fact, no events outside the forward elsewhere of the magic event has a nearby confederate. The forward elsewhere from any event is all the space-like events with positive position from that event bounded by light lines emanating from that event. An event near the magic event light trajectory although at finite times in the inertial coordinates is at plus or minus infinity in  $\tau$ . This feature of not being able to cover all of space time with confederates and bounded times will be intrinsic to accelerated coordinate systems and we will not be able to repair it. The infinite acceleration is problematic but not easy to overcome except to realize that these confederates are hypothetical.

A simpler coordinatizing scheme which was identical to the confederate method in the inertial case is achieved by using a protocol like the one in Section 11.1 on page 241 in which there is only one observer and that observer uses a clock and records the travel times of light to and from the event in question and then sets the coordinates as we did in the inertial case,

$$\begin{aligned} x &= \frac{c\tau_2 - c\tau_1}{2} \\ t &= \frac{\tau_2 + \tau_1}{2}. \end{aligned} \quad (14.23)$$

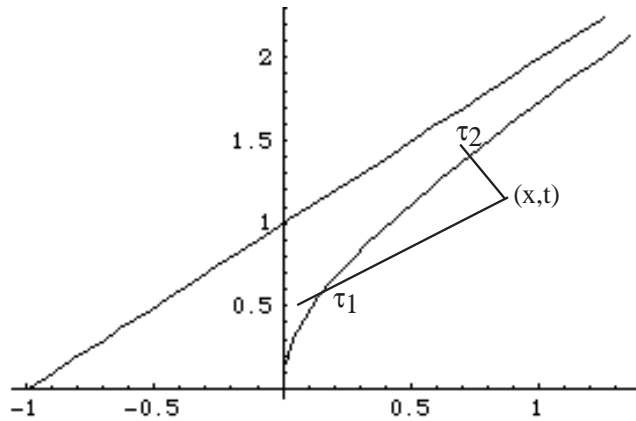


Figure 14.10: **Protocol for using an accelerated observer to coordinatize space-time.** The event that an inertial observer would label as  $(x_0, t_0)$  would be labeled as  $x = \frac{c\tau_2 - c\tau_1}{2}$  and  $t = \frac{\tau_2 + \tau_1}{2}$ .

This coordinatizing is shown in Figure 14.10 on page 325. This method of coordinatizing also has the advantage of not assuming that the underlying space is homogeneous. More will be made of this later, see Chapter 18 on page 373.

For a uniformly accelerated observer with acceleration  $g$  and setting the origin event at the zero velocity event of the observer, we can find the new coordinates,  $(x, t)$ , in terms of the inertial observers coordinates,  $(x_0, t_0)$ , by following the procedure in Section 11.2.3 on page 255 and Figure 11.7 on page 255. The equations of the two light cone lines from  $(x_0, t_0)$  are  $\frac{t-t_0}{x-x_0} = \pm \frac{1}{c}$ . Thus  $\tau_1$  and  $\tau_2$  satisfy

$$\frac{c^2}{g} \cosh\left(\frac{g\tau_1}{c}\right) - \frac{c^2}{g} - x_0 = \frac{c^2}{g} \sinh\left(\frac{g\tau_1}{c}\right) - ct_0$$

$$\frac{c^2}{g} \cosh\left(\frac{g\tau_2}{c}\right) - \frac{c^2}{g} - x_0 = -\frac{c^2}{g} \sinh\left(\frac{g\tau_2}{c}\right) + ct_0. \quad (14.24)$$

These can be solved for  $\tau_1$  and  $\tau_2$  and inserted into Equations 14.23 on page 325 to find  $(x, t)$ .

$$\begin{aligned} x &= \frac{c^2}{g} \ln \left[ \left(1 + \frac{(x_0 + ct_0)g}{c^2}\right) \left(1 + \frac{(x_0 - ct_0)g}{c^2}\right) \right] \\ t &= \frac{c}{g} \ln \left[ \frac{\left(1 + \frac{(x_0 + ct_0)g}{c^2}\right)}{\left(1 + \frac{(x_0 - ct_0)g}{c^2}\right)} \right]. \end{aligned} \quad (14.25)$$

Once again, note that these coordinates are singular on the light cone boundaries,  $-\frac{c^2}{g} = (x_0 \pm ct_0)$ , of the forward elsewhere from the magic event,  $(x_0 = -\frac{c^2}{g}, t_0 = 0)$ . In this coordinate, the range of  $x$  is  $-\infty < x < \infty$  and similarly for  $t$ . This looks more like a distance and a time. Despite this range in  $x$  and  $t$ , you should realize that this range of coordinates does not cover the entire range of  $(x_0, t_0)$  but only the forward elsewhere from the magic event. We can get a better feel for the shape of this coordinate system by removing those pesky ln functions. Redefining distance and time by

$$\begin{aligned} \eta &\equiv \exp\left(\frac{gx}{c^2}\right) \\ \zeta &\equiv \exp\left(\frac{gt}{c}\right). \end{aligned} \quad (14.26)$$

Plugging in and doing a little algebra,

$$\begin{aligned} \eta^2 \left(\frac{c^2}{g}\right)^2 &\equiv \left(x_0 + \frac{c^2}{g}\right)^2 - c^2 t_0^2 \\ \zeta^2 &\equiv \frac{1 + \frac{(x_0 + ct_0)g}{c^2}}{1 + \frac{(x_0 - ct_0)g}{c^2}}. \end{aligned} \quad (14.27)$$

Note that, in the forward elsewhere from the magic event,  $\eta^2$  and  $\zeta^2$  are positive with  $\eta$  equal to zero on both of the edges and  $\zeta$  equal to zero at the lower edge and plus infinity at the upper edge. From Equation 14.27 on page 326, it follows that events at the same distance, same  $x$  or  $\eta$ , are hyperbolas with the common magic event  $(-\frac{c^2}{g}, 0)$  in the inertial coordinate. In the new coordinate,  $(x, t)$ , the magic event is at spatial minus infinity or in  $(\eta, \zeta)$  at  $\eta = 0$ . The events at the same time, same  $t$  or  $\zeta$ , are straight

lines passing through the magic event. In the  $(x, t)$  coordinates, the lower edge is at minus infinity and the upper edge is at plus infinity. Thus this coordinate system looks like the system with confederates at fixed separation and adjusted accelerations, Figure 14.9 on page 324, with just a relabeling of distances and times. Obviously, lines of constant time,  $t$ , are lines of simultaneity to the special observer and the lines of fixed separation are the various suitably accelerated time-like curves. It is easy to show that this system of coordinatizing is the same as the one with the confederates with adjusted accelerations and with corrected clocks by merely reidentifying  $(h, \tau)$  in terms of  $(\eta, \zeta)$  or  $(x, t)$ .

$$\begin{aligned}\eta &= \frac{g}{c^2} \left( h + \frac{c^2}{g} \right) \\ \zeta &= \exp \left( \frac{g\tau}{c} \right)\end{aligned}\tag{14.28}$$

It is interesting to note that now that, although the relevant times are the same,  $t = \tau$ , the relevant distances are not the same,

$$h = \frac{c^2}{g} \left( \exp \left( \frac{gx}{c^2} \right) - 1 \right)\tag{14.29}$$

Confederates placed at equal spacing as measured in  $h$  will not be equally spaced in  $x$  even though the scale of length at the origin  $\Delta h$  and  $\Delta x$  are commensurate. At any place labeled by either  $h$  or  $x$ , the scales of distance are related by Equation 14.29 on page 327 and increments are related by

$$\Delta h = \exp \left( \frac{gx}{c^2} \right) \Delta x.\tag{14.30}$$

This is an example of a metric relationship. We will come upon this problem later in General Relativity, Section 17.7 on page 372. Which distance is the separation,  $h$  or  $x$ ? The  $\Delta h$  was constructed to be the proper distance between local confederates. The distance  $\Delta x$  is the incremental distance as measured by light travel time. Either can be used as the distance but practically speaking the light travel time method is the one that is utilized and thus makes sense as our measure although we will have to correct for the local distortion using the metric. This is one of the complications of accelerated systems.

We can complete the construction of our accelerated coordinate system in  $(x, t)$  by inverting Equation 14.25 on page 326,

$$x_0 = \frac{c^2}{g} \left( \exp \left( \frac{gx}{c^2} \right) \cosh \left( \frac{gt}{c} \right) - 1 \right)$$

$$t_0 = \frac{c}{g} \exp\left(\frac{gx}{c^2}\right) \sinh\left(\frac{gt}{c}\right). \quad (14.31)$$

Our interpretation of the distance measures can now be verified by using the metric that is provided by the inertial coordinate system. The interval, see Section 12.6 on page 295, between nearby events with differences in their coordinates of  $(\Delta x_0, \Delta t_0)$  is given by

$$\Delta x_{prop}^2 = \Delta x_0^2 - c^2 \Delta t_0^2 \quad (14.32)$$

where  $x_{prop}$  is the proper distance, if the separation is spacelike, and

$$\Delta \tau_{prop}^2 = \Delta t_0^2 - \frac{\Delta x_0^2}{c^2} \quad (14.33)$$

where  $t_{prop}$  is the proper time, if the separation is timelike. Using Equation 14.31 on page 328, these become

$$\Delta x_{prop}^2 = \exp\left(\frac{2gx}{c^2}\right) \{\Delta x^2 - c^2 \Delta t^2\}, \quad (14.34)$$

if the separation is spacelike, and

$$\Delta \tau_{prop}^2 = \exp\left(\frac{2gx}{c^2}\right) \left\{ \Delta t^2 - \frac{\Delta x^2}{c^2} \right\}, \quad (14.35)$$

if the separation is timelike. These same relations in the  $(h, \tau)$  coordinates are

$$\Delta x_{prop}^2 = \Delta h^2 - \left(h + \frac{c^2}{g}\right)^2 \frac{g^2}{c^2} \Delta \tau^2, \quad (14.36)$$

if the separation is spacelike, and a similar expression for the timelike case. Using the hangle, see Section 12.5 on page 289, between the magic event and the events in question,  $\Delta \phi \equiv \frac{g\Delta \tau}{c}$ , Equation 14.36 on page 328 becomes

$$\Delta x_{prop}^2 = \Delta h^2 - \left(h + \frac{c^2}{g}\right)^2 \Delta \phi^2. \quad (14.37)$$

The similarity between this form and the usual form for the distance in polar coordinates is striking and consistent with our interpretation of the hangle. See Figure 12.4 on page 277 and Figure ?? on page ??.

Can this system, particularly in  $(x, t)$ , generate a reasonable coordinate system? Will it? It should be obvious that that there are some serious problems here. Before we go into all the problems, lets look at how our

friend the accelerated observer would indicate events. Not thinking that he or she is particularly different, he/she would use a conventional grid for the labels of the events that are recorded. He/she would think that his/her measures of time and space are like those of an inertial observer and thus prepare an orthogonal grid to represent events. There is a clear and obvious

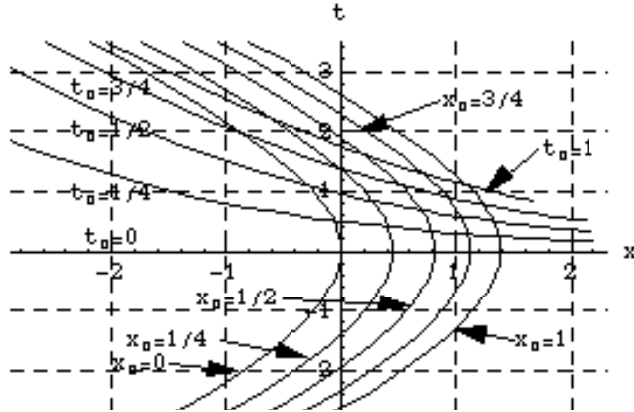


Figure 14.11: **Lines of Constant position and time in an accelerated coordinate system** In Figure 14.9 on page 324, the dashed lines represent events at either constant position, vertical dashed lines, or constant time, horizontal dashed lines, as designated by the inertial observer. In this figure these lines are the solid curves and the lines of constant position and time as designated by the accelerated observer are shown as dashed. Again in this figure lengths are in units of  $\frac{c^2}{g}$ .

distortion for the accelerated observer. Several features should be noted. It was noted above that, even though the range of position and time are the same as for the inertial observer, the events that are coordinatized are those in the forward light cone from the magic event and that points on these light lines, although finite to the inertial observer are mapped to infinity in these coordinates. In particular, note that the lines of constant  $t_0$  for  $t_0 = 1$  and  $x_0$  for  $x_0 = 0$  never cross and move off to  $\infty$  together. This is, of course, a reflection of the fact that the event  $(x_0 = 0, t_0 = 1)$  is on the light line from the magic event. Thus the accelerated observer thinks the all events are coordinatized but, as already discussed, the only events that can be coordinatized are in the forward elsewhere from the magic event. A further ramification is that, since lines of constant  $x_0$  are inertial and commoving with the accelerated observer at  $t = 0$ , these inertial observers experience a finite time between the events that bound the forward elsewhere from the

magic event and yet the accelerated observer says that this same observer experiences an infinite time interval between these events. Also note that, if the inertial observer should chose to pursue the inertial observer by accelerating in that direction, once the inertial observer passes the events bounding the forward elsewhere from the magic event, there is no acceleration that can accomplish this goal. This situation is very similar to the case of the black hole, see Section 18.1 on page 373, in which there is an event horizon and, in fact, the underlying physics is very similar.

All of these problems with the coordinatizing by the accelerated observer are also similar to those that emerge when attempting to coordinatize a curved space with a single flat map. Atlas maps of the earth are all distorted and some points such as the north pole are even topologically distorted, a point on the earth appears in the atlas as a line. As we will see in Section 17.7 on page 372, these similarities are not accidental.

## 14.6 Events at the Same Proper Time

Now consider the set events that are the same proper time from the origin event. These are the events that satisfy

$$c^2t^2 - x^2 = c\tau_0^2$$

where  $\tau_0$  is the proper time between the origin event and all the events on this curve.